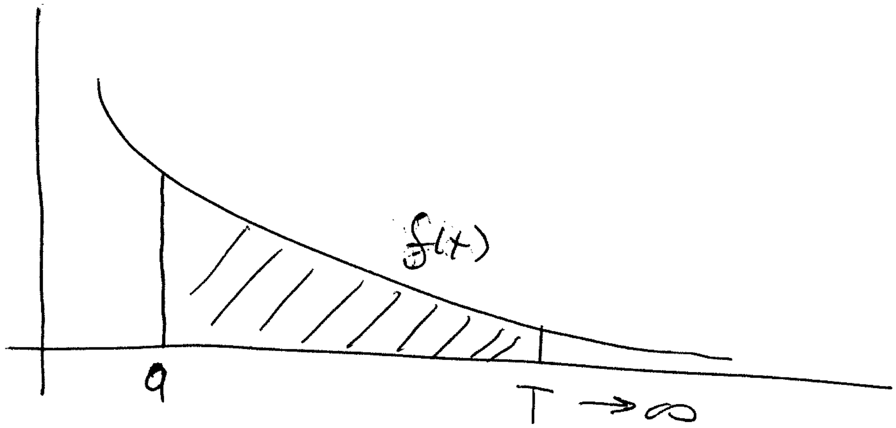
Type 1



$$\int_a^{\infty} f(x)\,dx = \lim_{T\to\infty} \int_a^{T} f(x)\,dx = K$$

→ If K is a finite number $\Rightarrow \int_a^{\infty} f(x)\,dx$ converges

→ If $K=\infty$ $\Rightarrow$ $\int_a^{\infty} f(x)\,dx$ diverges.

(area of the region is infinite)

$\boxed{p>0.}$

$\int_a^{\infty} \frac{dx}{x^p}$ converges if $p>1$, $a>0$ $\int_a^{\infty} \frac{dx}{x^2}$, $\int_3^{\infty} \frac{dx}{x^{\frac{125}{4}}}$, $\int_5^{\infty} \frac{dx}{x^{25}}$

$\int_a^{\infty} \frac{dx}{x^p}$ diverges if $p\le 1$, $a>0$ $\int_a^{\infty} \frac{dx}{\sqrt[15]{x^{13}}}$; $\int_a^{\infty} \frac{dx}{x^{1/3}}$; $\int_a^{\infty} \frac{dx}{x}$

$\int_0^{\infty} \frac{dx}{e^{\alpha x}}$ always converges, $(\alpha>0)$.

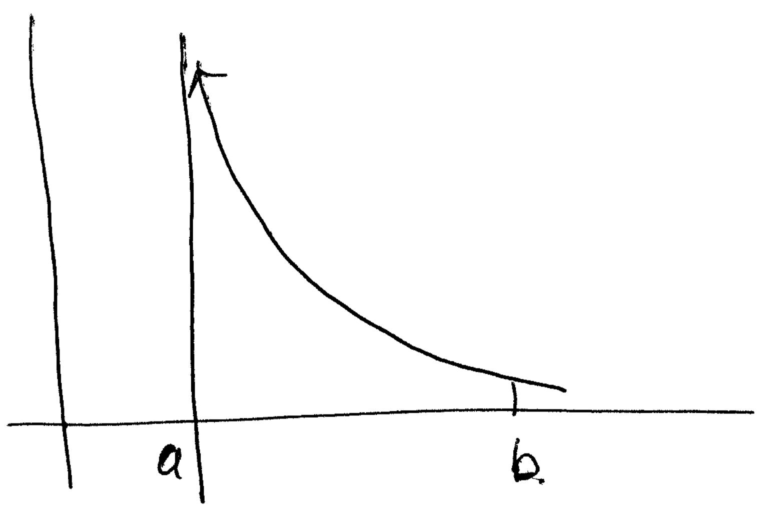
What is the value of the integral, $p>1$?

$$\int_1^\infty \frac{dx}{x^3} = \lim_{T\to\infty} \int_1^T \frac{dx}{x^3} = \lim_{T\to\infty} \left[\frac{x^{-2}}{-2} \Bigg|_{x=1}^{x=T} \right]$$

$\boxed{p=3}$

$$= \lim_{T\to\infty} \left[-\frac{1}{2T^2} + \frac{1}{2} \right] = \frac{1}{2}.$$

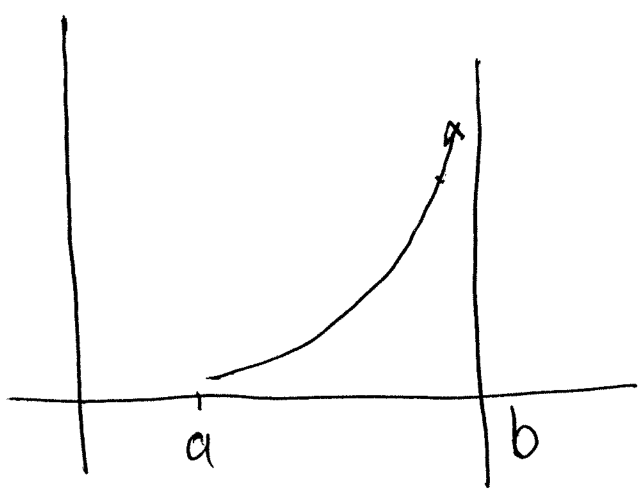
Type 2 (the integrand is unbounded)



$$\int_a^b f(x)\,dx$$

$f(x)$ is defined on $(a,b]$

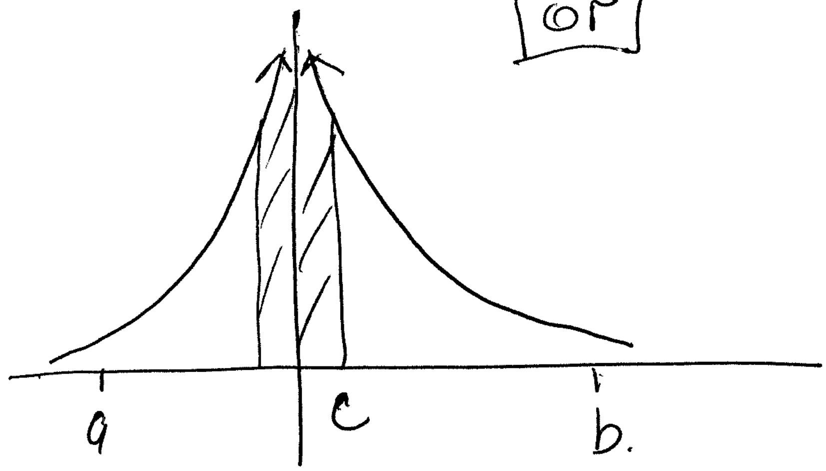
$\boxed{\text{or}}$



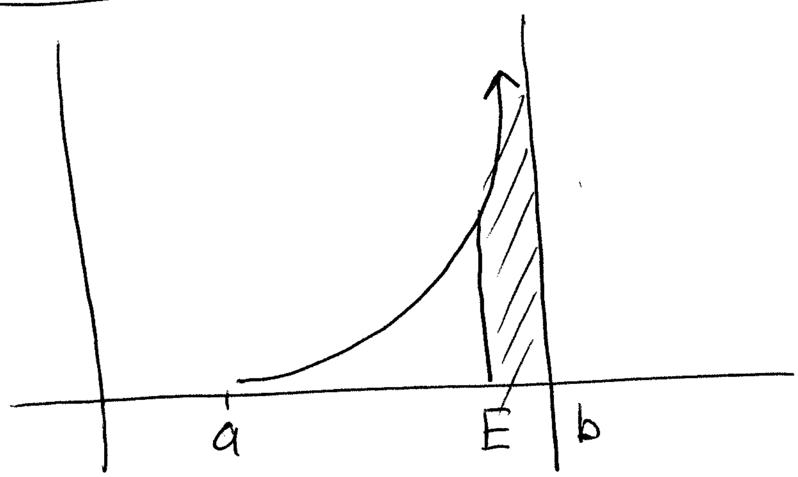
$$\int_a^b f(x)\,dx.$$

$f(x)$ is defined on $[a,b)$.

$\boxed{\text{or}}$



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$



$$\int_a^b f(x)dx = \lim_{E\to b^-} \int_a^E f(x)dx. = K$$

→ If K is a finite number $\Rightarrow$ $\int_a^b f(x)dx$ converges

→ If $K = \infty$ $\Rightarrow$ $\int_a^b f(x)dx$ diverges

(the area of the vertical region is infinite)

Look at $f(x) = \frac{1}{x^p}$ and $g(x) = \frac{1}{(x-a)^p}$,

where a is some number ~~a positive number~~, $p > 0$, $p \neq 1$.

Both of the integrals $\int_0^b \frac{dx}{x^p}$ and

$\int_a^b \frac{dx}{(x-a)^p}$ converge or diverge simultaneously.

Why?

We already know that $\int_0^b \frac{dx}{x^p}$ converges if

$p < 1$ and diverges if $p \geqslant 1$.

Let's make change of variable in the second integral:

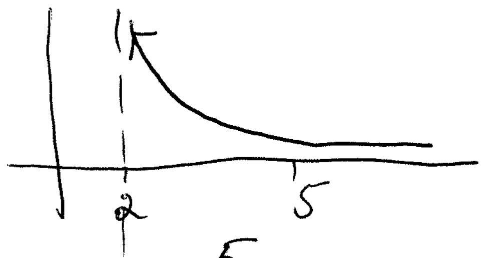
$$\boxed{\begin{array}{l} x - a = y \\ dx = dy \end{array} \qquad \begin{array}{c|c|c} x & a & b \\ \hline y & 0 & b-a \end{array}}$$

$b-a$ is still a finite number

Then $\int_a^b \frac{dx}{(x-a)^p} = \int_0^{b-a} \frac{dy}{y^p}$ $\leftarrow$ this integral is exactly of the form $\int_0^b \frac{dx}{x^p}$.

Thus, $\int_a^b \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geqslant 1$ ~~too~~.

Example $\displaystyle\int_2^5 \frac{dx}{\sqrt{x-2}} =$



$$= \lim_{E\to 2^+} \int_E^5 \frac{dx}{\sqrt{x-2}} = \lim_{E\to 2^+} \int_E^5 (x-2)^{-\frac{1}{2}}\,dx =$$

$$= \lim_{E\to 2^+} \left(\frac{(x-2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \Bigg|_{x=E}^{x=5} \right) = \lim_{E\to 2^+} \left(2(x-2)^{\frac{1}{2}} \Bigg|_E^5 \right) =$$

$$= \lim_{E\to 2^+} \left(2\cdot 3^{\frac{1}{2}} - 2(E-2)^{\frac{1}{2}} \right) = 2\cdot 3^{\frac{1}{2}} = \underline{\underline{2\sqrt{3}}}$$

the integral converges

or

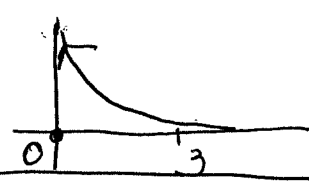
$$\int_2^5 \frac{dx}{\sqrt{x-2}} =$$

Substitution

$x-2 = u(x)$

$dx = du$

x	2	5
u	0	3



$=$

$p = \frac{1}{2}.$

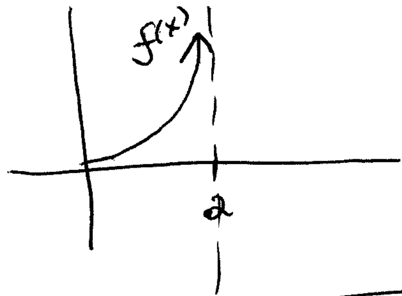
$$= \int_0^3 \frac{du}{u^{\frac{1}{2}}} = \lim_{\varepsilon\to 0^+} \int_\varepsilon^3 u^{-\frac{1}{2}}\,du = \lim_{\varepsilon\to 0^+} \left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right)\Bigg|_{u=\varepsilon}^{u=3} =$$

$$= \lim_{\varepsilon\to 0^+} \left(2u^{\frac{1}{2}}\right)\Big|_\varepsilon^3 = \underset{\varepsilon\to 0}{\left(2\cdot 3^{\frac{1}{2}} - 2\cdot\varepsilon^{\frac{1}{2}}\right)} = \underline{\underline{2\sqrt{3}}}$$

$\boxed{\text{Example}}$

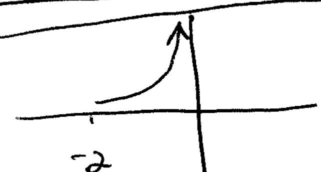
$$\int_0^2 \frac{dx}{(x-2)^2} = \text{type 2, } \underline{p=2>1}$$

since $f(x) = \dfrac{1}{(x-2)^2} \to \infty$ as $x \to 2^-$.



$$= \boxed{\begin{array}{l} \text{substitution: } \quad x-2 = u(x) \\ \qquad\qquad\qquad\quad\ dx = du \end{array} \quad \begin{array}{c|c|c} x & 0 & 2 \\ \hline u & -2 & 0 \end{array}}$$

$$= \int_{-2}^{0} \frac{du}{u^2} \quad \boxed{p>1}$$

= 

Shift to the left (by 2 units) =

$$= \lim_{\varepsilon \to 0^-} \int_{-2}^{\varepsilon} \frac{du}{u^2} = \lim_{\varepsilon \to 0^-} \left(-\frac{1}{u}\right)\Bigg|_{u=-2}^{u=\varepsilon} =$$

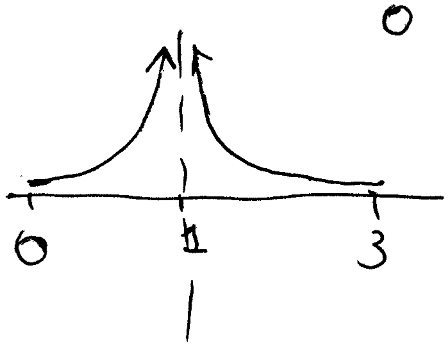
$$= \lim_{\varepsilon \to 0^-} \left(-\frac{1}{\varepsilon} + \frac{1}{-2}\right) \to +\infty, \text{ the integral diverges}$$

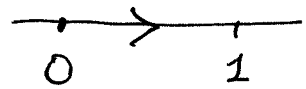
$\boxed{\text{or}}$

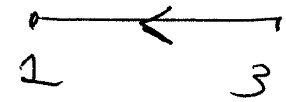
$$\int_0^2 \frac{dx}{(x-2)^2} = \lim_{E \to 2^-} \int_0^E \frac{dx}{(x-2)^2} = \lim_{E \to 2^-} \left(-\frac{1}{x-2}\right)\Bigg|_{x=0}^{x=E} =$$

$$= \lim_{E \to 2^-} \left(-\frac{1}{E-2} + \frac{1}{0-2}\right) \to +\infty, \text{ the integral diverges.}$$

Example $\displaystyle\int_0^3 \frac{dx}{x-1} = \underbrace{\int_0^1 \frac{dx}{x-1}}_{I_1} + \underbrace{\int_1^3 \frac{dx}{x-1}}_{I_2} =$







$$= \lim_{E\to 1^-} \int_0^E \frac{dx}{x-1} + \lim_{E\to 1^+} \int_E^3 \frac{dx}{x-1} =$$

$$= \lim_{E\to 1^-} \left(\ln|x-1|\right)\Big|_{x=0}^{x=E} + \lim_{E\to 1^+} \ln|x-1| \Big|_{x=E}^{x=3} =$$

$$= \lim_{E\to 1^-} \Big(\underbrace{\ln|E-1|}_{-\infty} - \underbrace{\ln 1}_{0}\Big) + \lim_{E\to 1^+} \Big(\ln 2 - \underbrace{\ln|E-1|}_{-\infty}\Big) =$$

The integral diverges.

Example $\displaystyle\int_0^\infty \frac{dx}{x^2} = \int_0^1 \frac{dx}{x^2} + \int_1^\infty \frac{dx}{x^2}$

$p = 2 > 1$

! $\boxed{a = 0}$

It is divergent   It is convergent

The original integral diverges as well

$\boxed{\text{Example}}$ $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{\ln x}{\sqrt{x}}\,dx =$

$$= \boxed{\begin{array}{ll} \ln x = u & \dfrac{dx}{\sqrt{x}} = dv = v'dx \\ \dfrac{dx}{x} = du & v = \displaystyle\int\frac{dx}{\sqrt{x}} = \int x^{-\frac{1}{2}}dx = 2\sqrt{x}. \end{array}}$$

$$= \lim_{\varepsilon\to 0^+}\left[2\sqrt{x}\cdot\ln x\Big|_{x=\varepsilon}^{x=1} - \int_\varepsilon^1 \frac{2\sqrt{x}\,dx}{x}\right] =$$

$$= \lim_{\varepsilon\to 0^+}\left[2\cdot 1\ln 1 - 2\sqrt{\varepsilon}\cdot\ln\varepsilon - \int_\varepsilon^1 \frac{2}{\sqrt{x}}\,dx\right] =$$

$$= \lim_{\varepsilon\to 0^+}\left[-2\varepsilon\ln\varepsilon - 4\sqrt{x}\Big|_\varepsilon^1\right] =$$

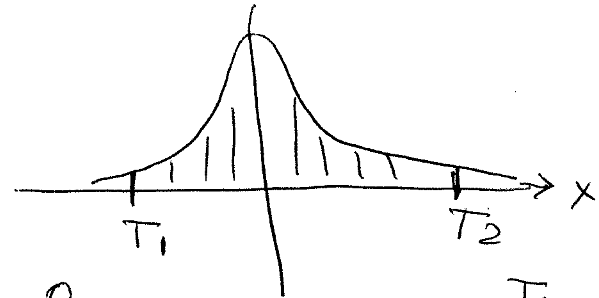
$$= \lim_{\varepsilon\to 0^+}\left[\cancel{-2\varepsilon\ln\varepsilon}^{\,0} - 4\cdot 1 + 4\cancel{\sqrt{\varepsilon}}^{\,0}\right] = -4.$$

? $\displaystyle\lim_{\varepsilon\to 0^+} \varepsilon\cdot\ln\varepsilon = [0\cdot(-\infty)] =$

$$= \lim_{\varepsilon\to 0^+}\frac{\ln\varepsilon}{\frac{1}{\varepsilon}} = \left[\frac{-\infty}{\infty}\right] = \lim_{\varepsilon\to 0^+}\frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} =$$

$$= \lim_{\varepsilon\to 0^+} -\frac{1}{\varepsilon}\cdot\frac{\varepsilon^2}{1} = \lim_{\varepsilon\to 0^+}(-\varepsilon) = 0. \text{ Thus,}$$

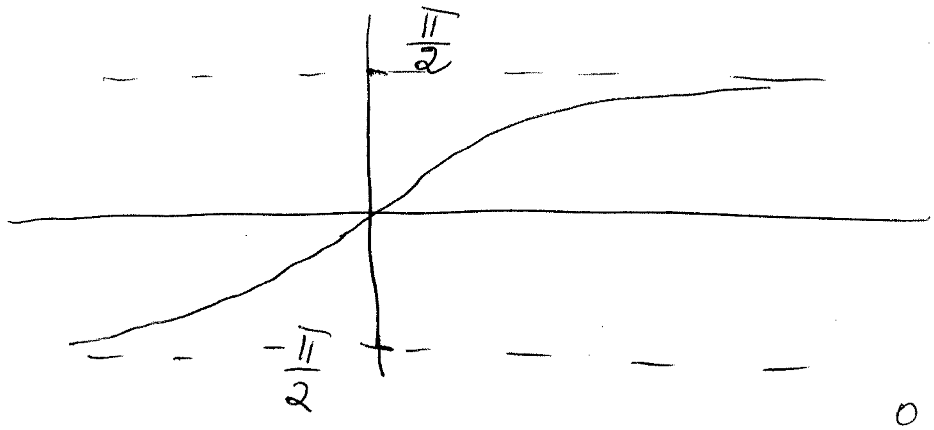
$\boxed{\text{Example}}$ $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} =$



$$= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \underbrace{\lim_{T_1 \to -\infty} \int_{T_1}^{0} \frac{dx}{1+x^2}}_{I_1} + \underbrace{\lim_{T_2 \to \infty} \int_{0}^{T_2} \frac{dx}{1+x^2}}_{I_2}$$

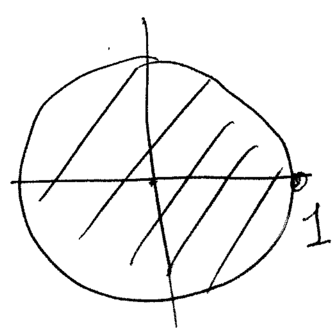
$$I_2 = \lim_{T_2 \to \infty} \int_{0}^{T_2} \frac{dx}{1+x^2} = \lim_{T_2 \to \infty} \left[\arctan x \Big|_{x=0}^{x=T_2} \right] =$$

$$= \lim_{T_2 \to \infty} \left[\arctan T_2 - \arctan 0 \right] = \frac{\pi}{2}.$$



Similarly, we can show that $\displaystyle\int_{-\infty}^{0} \frac{dx}{x^2} = \frac{\pi}{2}.$

Thus, area under the curve = area of unit circle



$A = \pi r^2 = \pi.$

$r = 1.$

Differential Equations are equations containing one or more derivatives of the unknown function.

For example, $\frac{d^2y}{dx^2} + \frac{dy}{dx} = xy.$ ← 2-nd order ODE

$\frac{dy}{dx} = xy.$ ← 1-st order ODE.

Many processes in physics, medicine, chemistry can be written mathematically using ODE model

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We want to describe the following situation:
Growth rate of a population at any time is proportional to the population size at that time.

Let $P(t)$ denote the size of the population at time t (number of individuals at time t).

Then $\frac{dP}{dt} = r\, P(t)$

r ← some constant of proportionality

We will restrict ourselves to ~~the~~ 1-st order differential equations of the form

$$\frac{dy}{dx} = f(x)\,g(y)$$

The RHS is the product of two functions; one function depends on x, the other one depends on y.

Such equations are called **separable dif. eq-ns**

Method to solve separable dif. eq-ns

We separate the variables x and y, so that one side of eqiation depends only on y, and the other side only on x:

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx$$

Then we integrate both sides, assuming $g(y) \neq 0$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

Separable Dif. Equations

include two special cases

$$\frac{dy}{dx} = f(x) \quad [g(y) = 1]$$

Pure time Dif. Eq-ns

$$\frac{dy}{dt} = t^2; \quad \frac{dy}{dx} = \sin x$$

$y = y(t)$; $y = y(x)$

$$\frac{dy}{dx} = g(y)$$

$$[f(x) = 1]$$

Autonomous Dif. Eq-ns

$$\frac{dy}{dx} = y^2 \ ; \ y = y(x)$$

$$\frac{dy}{dt} = y(y-1) \ ; \ y = y(t)$$

Example

Suppose that a tree grows according to the equation

$$\frac{dH}{dt} = 7e^{-0.2t}$$

$H(0) = 50$ (cm) ← Pure-time Dif. eq.

(a) Find the general solution to the equation

$$dH = 7e^{-0.2t}\,dt.$$

$$\int dH = \int 7e^{-0.2t}\,dt$$

$$H(t) = 7\int e^{-0.2t}\,dt = \frac{7e^{-0.2t}}{-0.2} + C =$$

$$= -35e^{-0.2t} + C.$$

$$\underbrace{H(t) = -35e^{-0.2t} + C}$$

$$\int dH(t) = \int 7e^{-0.2t}\,dt$$

$$H(t) = 7\int e^{-0.2t}\,dt = \frac{7e^{-0.2t}}{-0.2} + C =$$

$$= -35e^{-0.2t} + C.$$

(b) Find the particular solution of the equation corresponding to the initial condition $H(0) = 50\,(cm)$

$$H(t) = -35e^{-0.2t} + C$$

$$H(0) = -35e^{-0.2\cdot 0} + C = -35 + C = 50 \Rightarrow$$

$$\boxed{C = 85}$$

$$\boxed{H(t) = -35e^{-0.2t} + 85}$$

(c) Determine the height of the tree after 5 years ($H(5)$) and after 7 years ($H(7)$)

How much did the tree grow b/n $t=7$ and $t=5$?

$$H(5) = -35e^{-0.2\cdot 5} + 85 = -\frac{35}{e} + 85 = 72.08\,(cm)$$

$$H(7) = -35e^{-1.4} + 85 = 76.3\,(cm)$$

$$H(7) - H(5) = 4.25\,(cm).$$

(e) Determine how much did the tree grow between $t=5$ and $t=7$ using the FTC

$$\int_5^7 H'(t)\,dt = H(t)\Big|_{t=5}^{t=7} = \overbrace{H(7)-H(5)}$$

$$\int_5^7 7e^{-0.2t}\,dt = \frac{7e^{-0.2t}}{-0.2}\Big|_{t=5}^{t=7} =$$

$$= -35e^{-0.2t}\Big|_{t=5}^{t=7} = -35e^{-1.4} + 35e^{-1} =$$

$$= 4.247(\text{cm}) = \underbrace{H(7)-H(5)}$$

how much did the tree grow b/n $t=5$ and $t=7$

The FTC

$$\boxed{\int_a^b f(t)\,dt = F(b)-F(a), \quad F'(t) = f(t)}$$